New Approach for the Analysis of Flexure of Symmetric Laminates

K. Jawahar Reddy,* K. S. Nagaraja,† and K. Vijayakumar‡
Indian Institute of Science, Bangalore 560 012, India

In this paper, a refined shear-deformation theory for flexure of thick symmetric angle-ply laminates has been developed. Lamination-dependent thicknesswise distributions for in-plane displacements in terms of reference-plane variables are used. They are derived from a priori distributions for slopes of transverse shear stresses based on three-dimensional equilibrium equations in each ply. Performance of the present theory is evaluated by comparing the solutions with the exact elasticity solutions for simply supported cross-ply laminates subjected to sinusoidal loading. It is found that the present theory is quite accurate for a wide range of variations in the thickness and modular ratios.

Introduction

EXTENSIVE use of fiber-reinforced composites in engineering structures has evoked considerable interest in the accurate prediction of response characteristics of laminates, particularly, interlaminar stresses. Directional nature of composites and relative magnitudes of moduli render secondary effects such as transverse stresses more pronounced. In spite of idealizing each ply as orthotropic, the complexity of three-dimensional elasticity approach has resulted in the development of several two-dimensional approximate models (for an annotated bibliography, see Refs. 1 and 2). Most of these models may be broadly classified into the following three groups:

Group I

Models based on thicknesswise ply-independent displacement distributions (these models are referred as single-layer theories in Ref. 3).

Group II

Discrete layer models.[4–9]

Group III

Models based on a priori derived lamination-dependent displacement distributions.[10–16]

Group I models consist of a power-series expansion of displacements (u,v,w) in terms of the thicknesswise coordinate z. In these models, all strains are continuous throughout the laminate. There is no provision for the lamination-dependent discontinuities of slopes $(u_{,z},v_{,z},w_{,z})$ across interfaces. As such, transverse stresses from constitutive relations display unrealistic discontinuities across interfaces. Consequently, the convergence of direct estimates to transverse strains and stresses is prohibitively slow, particularly along interfaces. This problem, in practice, is overcome by taking recourse to integration of equilibrium equations to estimate transverse stresses and obtain transverse strains using constitutive relations.

In group II models, in-plane variables are introduced at the ply level. As such, the number of variables in each of these models depend on the number of plies. Lamination effects in the thicknesswise distributions of displacements are accounted for through the in-plane variables only, and the assumed ply distributions are not dependent on ply material constants. In earlier moels,[4,5] in-plane variations of displacements are introduced at the middle plane of each ply. Governing equations for the determination of these displacements involve continuity conditions for transverse stresses across interfaces. In some later models,[6,7] these variations of displacements are introduced at interfaces and interconnected by using interpolation functions through the thickness of each ply. Although these models are convenient for finite element implementation, the transverse stresses through constitutive relations are discontinuous across interfaces and one has to take recourse to statically equivalent estimates such as in group I models. A more sophisticated model is due to Pagano[8] and it consists of introducing, along with displacements, interface transverse shear stresses as variables and deriving the governing equations through Reissner's variational principle. Application of the model in practical situations is simplified by adopting a local-global approach outlined in a later investigation.[9]

Models in group III retain the simplicity of group I models in using reference-plane variables (so that they are independent of the number of plies) but take into account material discontinuities. Lamination effects are reflected, unlike in group II models, by a priori defined thicknesswise displacement distributions. These distributions are derived from assumed/derived transverse normal strain (however, it is generally assumed to be zero) and transverse shear stresses.

It appears that the concept of deriving lamination-dependent displacement distributions from assumed transverse shear stresses was proposed, about two decades ago, by Ambartsumyan[10] for cross-ply laminates and extended to angle-ply laminates by Whitney.[11]

A systematic approach to derive lamination-dependent displacement distributions was presented by Valisetty and Rehfield,[12] in which statically equivalent transverse shear stresses and transverse normal strain from constitutive relations expressed in terms of classical plate theory (CPT) stress resultants were used. Later, Vijayakumar and Krishna Murty[13] adopted the iterative concept inherent in the above approach and outlined a procedure to derive displacement distributions in terms of reference-plane displacement variables and their derivatives.

An alternative formulation of in-plane displacement distributions in terms of reference-plane variables accounting for material discontinuities was proposed by Ren.[14] He assumed thicknesswise distributions for transverse shear stresses by using the exact solution of a cantilever laminated infinite strip subjected to edge loads. The displacement distributions, however, 1) lack independent in-plane variations in the higher-order stretching terms, and 2) do not reflect the influence of in-plane shear modulus in flexure terms.

Received July 25, 1990; revision received Jan. 3, 1991; accepted for publication Jan. 3, 1991.

*Research Student, Department of Aerospace Engineering.
†Scientific Assistant, Department of Aerospace Engineering.
‡Associate Professor, Department of Aerospace Engineering.

Recently, we have proposed a new approach to generate several models of the group III type.[15] This approach is based on assumed plywise distributions in terms of reference-plane variables for transverse normal strain ε_z and for slopes of transverse shear stresses $\sigma_{4,z}$ and $\sigma_{5,z}$.

In the case of flexure of symmetric laminates, the simplest shear-deformation model in the aforementioned category is derived from assumed zero values for $\sigma_{4,z}$ and $\sigma_{5,z}$. The resulting model replaces the thicknesswise constant transverse shear strains in first-order shear-deformation theory (FSDT) by constant transverse shear stresses. The model can be shown to be equivalent to an earlier model corresponding to theory II presented in their work by Sun and Whitney.[5] A refined model is obtained by assuming a linear variation of $\sigma_{4,z}$ and $\sigma_{5,z}$ through the thickness of the laminate in view of their antisymmetric nature with respect to the middle plane. An equivalent model has been recently proposed independently by Lee et al.[16] In the above models, the thicknesswise shear-stress distributions are independent of ply material properties. The derived displacement distributions are, however, dependent on transverse shear compliances.

Other group III models mentioned earlier can be interpreted by the above approach. Confining to shear-deformation models for flexure of symmetric laminates, the displacement distributions in these models can be derived from the plywise distributions of $\sigma_{4,z}$ and $\sigma_{5,z}$ as given in Table 1.

In the present work, three-dimensional elasticity equilibrium equations in each ply are considered. The in-plane stresses in these equations are expressed in terms of CPT in-plane strains. For flexure of symmetric angle-ply laminates, each of the expressions for $\sigma_{4,z}$ and $\sigma_{5,z}$ contain all of the four third-order partial derivatives of transverse displacement $w_o(x,y)$. Based on the above, we assume a linear variation of $\sigma_{4,z}$ and $\sigma_{5,z}$ in terms of four unknown variables associated with appropriate ply material constants (see Table 1). Using these distributions along with $w_o(x,y)$, the expressions for in-plane displacements are derived.

Here, the energy method (principle of virtual displacements) is suggested to determine the unknown variables in the aforementioned expressions for displacements. Extensive numerical work has been carried out in order to assess the performance of the present procedure by comparison with the exact analysis of simply supported cross-ply laminated plates subjected to sinusoidal loading.

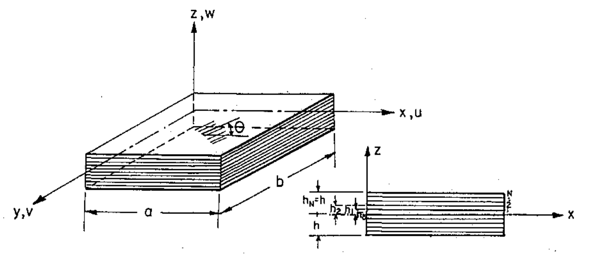

Fig. 1 Typical laminated plate: geometry and coordinates.

Formulation

Figure 1 shows a typical $2N$-layered symmetric laminated composite plate. The thickness $2h$ of the plate is small compared to its lateral dimensions a and b. The plate is subjected to an asymmetric normal load $q(x,y)$ along top and bottom surfaces ($z = \pm h$).

Constitutive Equations

The constitutive equations of an orthotropic lamina in the laminate coordinate system are, in the usual contracted notation,

$$\sigma_i = \overline{Q}_{ij}\varepsilon_j, \qquad i = 1,2,6 \tag{1a}$$

$$\varepsilon_\alpha = \overline{S}_{\alpha\beta}\sigma_\beta, \qquad \alpha = 4,5 \tag{1b}$$

$$\varepsilon_3 = \overline{S}_{33}\sigma_3 + \overline{S}_{3j}\sigma_j \tag{1c}$$

where $\overline{Q}_{ij}$ are transformed plane stress-reduced elastic constants and $\overline{S}_{3j}$, $\overline{S}_{\alpha\beta}$ are transformed compliance elastic constants. In the above relations, the repeat index in a term indicates summation over its range of specified integer values; suffix j takes values of 1, 2, and 6, and β takes values of 4 and 5.

Expressions for In-Plane Displacements

In the present procedure, the ply-dependent expressions for in-plane displacements are obtained from consideration of three-dimensional elasticity equilibrium equations. For this purpose, we consider, first, the displacements for flexure of symmetric laminates in the classical plate theory given by

$$w = w_o \tag{2a}$$

$$u = -zw_{o,x} \tag{2b}$$

$$v = -zw_{o,y} \tag{2c}$$

Using the strain-displacement relations

$$\varepsilon_1 = u_{,x}, \qquad \varepsilon_2 = v_{,y}, \qquad \varepsilon_6 = u_{,y} + v_{,x} \tag{3}$$

and constitutive relations Eq. (1a), the two in-plane equilibrium equations in each ply,

$$\sigma_{1,x} + \sigma_{6,y} + \sigma_{5,z} = 0 \tag{4a}$$

$$\sigma_{6,x} + \sigma_{2,y} + \sigma_{4,z} = 0 \tag{4b}$$

Table 1 Assumed $\sigma_{4,z}$ and $\sigma_{5,z}$ distributions in each ply for flexure of symmetric laminates

Model/theory	$\sigma_{5,z}$	$\sigma_{4,z}$
Ambartsumyan[10]	$f'(z)\overline{Q}_{55}\psi_1$	$f'(z)\overline{Q}_{44}\psi_2$
Whitney[11]	$f'(z)(\overline{Q}_{55}\psi_1 + \overline{Q}_{45}\psi_2)$	$f'(z)(\overline{Q}_{45}\psi_1 + \overline{Q}_{44}\psi_2)$
Valisetty and Rehfield[a]	$z\{D_{1j}M_{j,x} + D_{6j}M_{j,y}\}$	$z\{D_{6j}M_{j,x} + D_{2j}M_{j,y}\}$
Ren[14]	$z\{\overline{Q}_{11}\psi_1 + \overline{Q}_{26}\psi_2 + (\overline{Q}_{11}\overline{Q}_{22}/\overline{Q}_{12})\psi_3\}$	$z\{\overline{Q}_{16}\psi_1 + \overline{Q}_{22}\psi_2 + (\overline{Q}_{11}\overline{Q}_{22}/\overline{Q}_{12})\psi_4\}$
Present model	$z\{\overline{Q}_{11}\psi_1 + (\overline{Q}_{12} + 2\,\overline{Q}_{66})\psi_2 + \overline{Q}_{26}\psi_3 + 3\overline{Q}_{16}\psi_4\}$	$z\{\overline{Q}_{16}\psi_1 + 3\,\overline{Q}_{26}\psi_2 + \overline{Q}_{22}\psi_3 + (\overline{Q}_{12} + 2\overline{Q}_{66})\psi_4\}$

[a]The two first-order partial derivatives of each of the stress resultants M_x, M_y, M_{xy} are used and $D^k_{ij} = C^k_{i}, D^*_{sj}$, where $[C^k]$ is the plane stress stiffness matrix for the kth ply and D^* denotes bending flexibility matrix as defined in Ref. 12.

take the form

$$\sigma_{4,z} - z[\overline{Q}_{16}w_{o,xxx} + 3\overline{Q}_{26}w_{o,xyy} + \overline{Q}_{22}w_{o,yyy} + (\overline{Q}_{12} + 2\overline{Q}_{66})w_{o,xyy} = 0 \tag{5a}$$

$$\sigma_{5,z} - z[\overline{Q}_{11}w_{o,xxx} + (\overline{Q}_{12} + 2\overline{Q}_{66})w_{o,xyy} + \overline{Q}_{26}w_{o,yyy} + 3\overline{Q}_{16}w_{o,xyy} = 0 \tag{5b}$$

Now replacing the four third-order derivatives of w_o in the above equations by four unknown variables ψ_i (i = 1,2,3, and 4), the slopes of transverse shear stresses in the kth ply are expressed as

$$\sigma_{4,z} = z[\overline{Q}_{16}\psi_1 + 3\overline{Q}_{26}\psi_2 + \overline{Q}_{22}\psi_3 + (\overline{Q}_{12} + 2\overline{Q}_{66})\psi_4] \tag{6a}$$

$$\sigma_{5,z} = z[\overline{Q}_{11}\psi_1 + (\overline{Q}_{12} + 2\overline{Q}_{66})\psi_2 + \overline{Q}_{26}\psi_3 + 3\overline{Q}_{16}\psi_4] \tag{6b}$$

Introducing the ply coordinate

$$z_k = h_k - z \tag{7}$$

Eqs. (6) are more conveniently written as

$$\frac{\partial\sigma_\alpha}{\partial z_k} = f_o\, \tilde{Q}_{\alpha m}\psi_m, \qquad \alpha = 4,5 \tag{8}$$

where the repeat index m takes values 1 to 4, and,

$$f_o = (z_k - h_k) \tag{9}$$

$$\begin{aligned} \tilde{Q}_{41} &= \overline{Q}_{16}, & \tilde{Q}_{51} &= \overline{Q}_{11} \\ \tilde{Q}_{42} &= 3\overline{Q}_{26}, & \tilde{Q}_{52} &= (\overline{Q}_{12} + 2\overline{Q}_{66}) \\ \tilde{Q}_{43} &= \overline{Q}_{22}, & \tilde{Q}_{53} &= \overline{Q}_{26} \\ \tilde{Q}_{44} &= (\overline{Q}_{12} + 2\overline{Q}_{66}), & \tilde{Q}_{54} &= 3\overline{Q}_{16} \end{aligned} \tag{10}$$

By integrating Eqs. (8) and using interface continuity and shear-free surface conditions, the expressions for transverse shear stresses in the kth ply are obtained as

$$\sigma_\alpha = [(B_{\alpha m})_N - (B_{\alpha m})_k + \tilde{Q}_{\alpha m}\, f_{o1}]\psi_m, \qquad \alpha = 4,5 \tag{11}$$

where

$$(B_{\alpha m})_k = \sum_{l=1}^{k} (\tilde{Q}_{\alpha m})_1\, (t_l^2/2 - h_l t_l) \tag{12a}$$

$$f_{o1} = (z_k^2/2 - h_k z_k) \tag{12b}$$

Substituting the above expressions for σ_α in the constitutive relations (1b), the transverse shear strains ε_α are expressed as

$$\varepsilon_\alpha = [\tilde{B}_m^{(\alpha)} + \mu_m^{(\alpha)}\, f_{o1}]\psi_m, \qquad \alpha = 4,5 \tag{13}$$

where

$$\tilde{B}_m^{(\alpha)} = \overline{S}_{\alpha\beta}[(B_{\beta m})_N - (B_{\beta m})_k] \tag{14a}$$

$$\mu_m^{(\alpha)} = \overline{S}_{\alpha\beta}\tilde{Q}_{\beta m} \tag{14b}$$

Using the above strains [Eqs. (13)] along with assumed normal strain $\varepsilon_3 = 0$ in the transverse strain-displacement relations

$$\varepsilon_3 = w_{,z} \tag{15a}$$

$$\varepsilon_4 = w_{,y} + v_{,z} \tag{15b}$$

$$\varepsilon_5 = w_{,x} + u_{,z} \tag{15c}$$

and integrating, one obtains displacements given below in each ply with the zero reference-plane values for in-plane displacements due to symmetry and maintaining continuity across interfaces:

$$w = w_o(x,y) \tag{16a}$$

$$u = f_o\, w_{o,x} + [D_m^{(1)} - f_m^{(1)}]\psi_m \tag{16b}$$

$$v = f_o\, w_{o,y} + [D_m^{(2)} - f_m^{(2)}]\psi_m \tag{16c}$$

where

$$f_m^{(1)} = z_k\tilde{B}_m^{(5)} + \mu_m^{(5)}\, f_{o2} \tag{17a}$$

$$f_m^{(2)} = z_k\tilde{B}_m^{(4)} + \mu_m^{(4)}\, f_{o2} \tag{17b}$$

$$f_{o2} = z_k^3/6 - h_k z_k^2/2 \tag{17c}$$

$$D_m^{(p)} = \sum_{l=1}^{k} [f_m^{(p)}]_{z_l = t_l}, \qquad p = 1,2 \tag{17d}$$

Variational Formulation

Replacing u and v by u_1 and u_2, respectively, and introducing $L_1 = \partial/\partial x$ and $L_2 = \partial/\partial y$, the expressions for the in-plane displacements are rewritten as

$$u_p = f_o L_p w_o + g_m^{(p)}\psi_m, \qquad p = 1,2 \tag{18}$$

where

$$g_m^{(p)} = (D_m^{(p)} - f_m^{(p)}) \tag{19}$$

From the displacements [Eqs. (18)] and the strain-displacement relations

$$\varepsilon_i = L_i^{(p)}\, u_p, \qquad i = 1,2,6 \tag{20}$$

where

$$L_1^{(1)} = L_1, \qquad L_2^{(1)} = 0, \qquad L_6^{(1)} = L_2 \tag{21a}$$

$$L_1^{(2)} = 0, \qquad L_2^{(2)} = L_2, \qquad L_6^{(2)} = L_1 \tag{21b}$$

the in-plane strains are given by

$$\varepsilon_i = f_o L_{io} w_o + g_m^{(p)}\, L_i^{(p)}\psi_m, \qquad i = 1,2,6 \tag{22}$$

where

$$L_{1o} = L_1(L_1), \qquad L_{2o} = L_2(L_2), \qquad L_{6o} = 2\, L_1(L_2) \tag{23}$$

The strain energy of the laminate is given by

$$U = \sum_{k=1}^{N} \int_V (\sigma_i\varepsilon_i + \sigma_\alpha\varepsilon_\alpha)\, \mathrm{d}V \tag{24}$$

$$= \sum_{k=1}^{N} \int_V \{\overline{Q}_{ij}[f_o L_{jo} w_o + g_m^{(p)} L_j^{(p)}\psi_m][f_o L_{io} w_o + g_n^{(q)} L_i^{(q)}\psi_n] + \overline{S}_{\alpha\beta}[(B_{\alpha m})_N - (B_{\alpha m})_k + \tilde{Q}_{\alpha m} f_{o1}][(B_{\beta n})_N - (B_{\beta n})_k + \tilde{Q}_{\beta n} f_{o1}]\psi_m\psi_n\}\, \mathrm{d}V \tag{25}$$

After carrying out thicknesswise integrations, the above expression for the strain energy takes the form

$$
\begin{aligned}
U = \sum_{k=1}^{N} \int_A & \{(\bar{Q}_{ij})_k[I_1 L_{io} w_o L_{jo} w_o + I^{(p)}_{om} L^{(p)}_j \psi_m L_{io} w_o \\
& + I^{(q)}_{on} L^{(q)}_i \psi_n L_{jo} w_o + I^{(p,q)}_{mn} L^{(p)}_j \psi_m L^{(q)}_i \psi_n] \\
& + (\bar{S}_{\alpha\beta})_k \{t_k[(B_{\alpha m})_N - (B_{\alpha m})_k]\,[(B_{\beta n})_N - (B_{\beta n})_k] \\
& + I_2\{\tilde{Q}_{\beta n}[(B_{\alpha m})_N - (B_{\alpha m})_k] + \tilde{Q}_{\alpha m}[(B_{\beta n})_N \\
& - (B_{\beta n})_k]\} + I_3 \tilde{Q}_{\alpha m}\, \tilde{Q}_{\beta n}\}\psi_m \psi_n\}\, \mathrm{d}A
\end{aligned}
\tag{26}
$$

where

$$(I_1, I_2, I_3) = \int_o^{t_k} [(f_o)^2, f_{o1}, (f_{o1})^2]\, \mathrm{d}z_k \tag{27a}$$

$$I^{(p)}_{om} = \int_o^{t_k} f_o g^{(p)}_m \, \mathrm{d}z_k \tag{27b}$$

(with a similar expression for $I^{(q)}_{on}$) and

$$I^{(p,q)}_{mn} = \int_o^{t_k} g^{(p)}_m g^{(q)}_n \, \mathrm{d}z_k \tag{27c}$$

Work done W by the applied normal load is given by

$$W = 2 \int_A q(x,y) w_o(x,y)\, \mathrm{d}A \tag{28}$$

The principle of virtual displacements $\delta\,(U - W) = 0$ may now be applied to derive the necessary differential equations and boundary conditions or to generate algebraic equations of a finite element scheme to determine the unknown variables w_o and $\psi_m (m = 1,2,3,$ and $4)$.

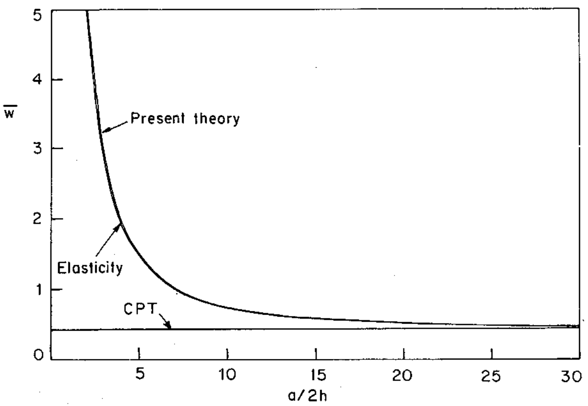

Fig. 2 Nondimensionalized central-plane deflection $\bar{w}$ vs $a/2h$.

Statically Equivalent Transverse Shear Stresses

As in all shear-deformation models based on the principle of virtual displacements, there is a need to estimate the interlaminar stresses through the integration of equilibrium equations, particularly along a clamped edge. For this purpose, the in-plane equilibrium equations in each ply, in contracted notation, are written as

$$\frac{\partial \sigma_\alpha}{\partial z_k} = L^{(i)}_\alpha\, \sigma_i, \qquad \alpha = 4,5 \tag{29}$$

where

$$
\begin{aligned}
L^{(1)}_4 &= 0, \qquad L^{(2)}_4 = L_2, \qquad L^{(6)}_4 = L_1 \\
L^{(1)}_5 &= L_1, \qquad L^{(2)}_5 = 0, \qquad L^{(6)}_5 = L_2
\end{aligned}
\tag{30}
$$

Integration of Eqs. (29), using 1) constitutive relations (1a) for σ_i; 2) in-plane strains given by Eq. (20); 3) interface continuity conditions; and 4) shear-free surface conditions, leads to the expressions given below for the transverse shear stresses.

$$
\begin{aligned}
\sigma_\alpha = {} & [(B_{ij})_N - (B_{ij})_k + \bar{Q}_{ij} f_{o1}]\, L^{(i)}_\alpha L_{jo} w_o \\
& + [(F^{(p)}_{ijm})_N - (F^{(p)}_{ijm})_k + g^{(p)}_{m1} \bar{Q}_{ij}] L^{(i)}_\alpha L^{(p)}_j \psi_m
\end{aligned}
\tag{31}
$$

$$(B_{ij})_k = \sum_{l=1}^{k} \bar{Q}_{ij}(t_l^2/2 - h_l t_l) \tag{32a}$$

$$(F^{(p)}_{ijm})_k = \sum_{l=1}^{k} \bar{Q}_{ij}(g^{(p)}_{m1})_{z_l = t_l} \tag{32b}$$

$$g^{(p)}_{m1} = \int_o^{t_k} g^{(p)}_m \, \mathrm{d}z_k \tag{33}$$

Numerical Results and Discussion

A simply supported, square, symmetric, cross-ply laminate subjected to sinusoidal load

$$(\sigma_3)_{z \pm h} = \pm q_o/2 \sin\pi x/a \sin\pi y/a \tag{34}$$

is considered for the numerical investigations. The solutions for the displacements take the form

$$w = \bar{w} \sin\pi x/a \sin\pi y/a \tag{35a}$$

$$u = \bar{u}(z) \cos\pi x/a \sin\pi y/a \tag{35b}$$

$$v = \bar{v}(z) \sin\pi x/a \cos\pi y/a \tag{35c}$$

Numerical data are generated both by the present procedure and the exact analysis given by Pagano[17] considering the following material properties, with material axes L and T along and perpendicular to fiber direction, respectively.

Table 2 Stresses in a square $[0/90]_S$ laminate ($E_L/E_T = 25$)

	$(\bar{\sigma}_1)_{z=h}$		$(\bar{\sigma}_2)_{z=h/2\ (90\ \mathrm{deg})}$		$(\bar{\sigma}_6)_{z=h}$		$(\bar{\sigma}_5)_{z=0}$		$(\bar{\sigma}_4)_{z=0}$	
$a/2h$	Elasticity	PT[a]	Elasticity	PT	Elasticity	PT	Elasticity	PT	Elasticity	PT
4	0.702	0.737	0.664	0.674	−0.046	−0.048	0.219	0.218	0.292	0.290
10	0.559	0.562	0.402	0.402	−0.028	−0.028	0.301	0.302	0.196	0.196
20	0.543	0.544	0.309	0.309	−0.023	−0.023	0.328	0.328	0.156	0.155
50	0.539	0.539	0.276	0.276	−0.022	−0.022	0.337	0.337	0.141	0.141
100	0.539	0.539	0.271	0.271	−0.021	−0.021	0.339	0.339	0.139	0.139
	Classical plate theory									
——	0.539		0.269		−0.021		0.339		0.138	

[a]Present theory.

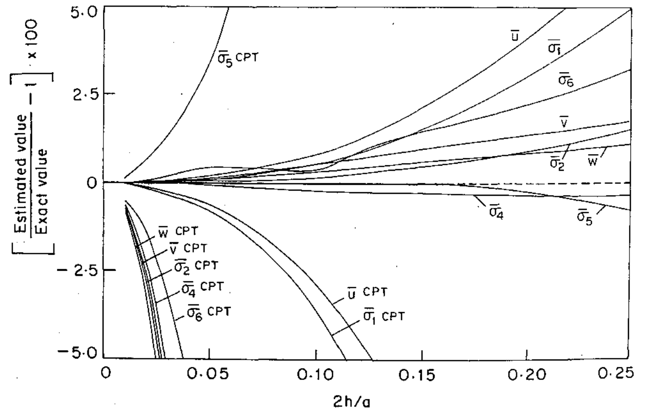

Fig. 3 Percentage errors in the estimates to displacements and stresses vs $2h/a$ (E_L/E_T = 25).

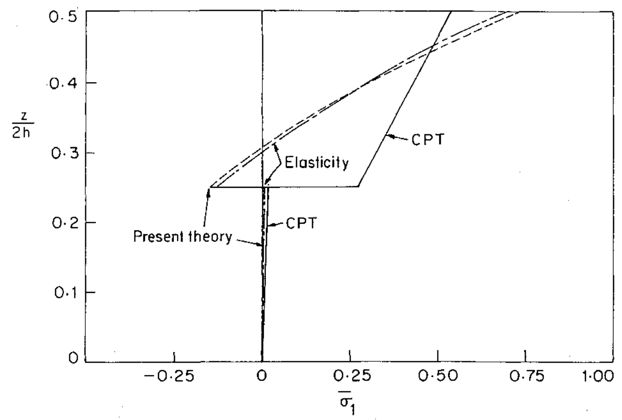

Fig. 6 Nondimensionalized in-plane normal stress $\bar{\sigma}_1$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

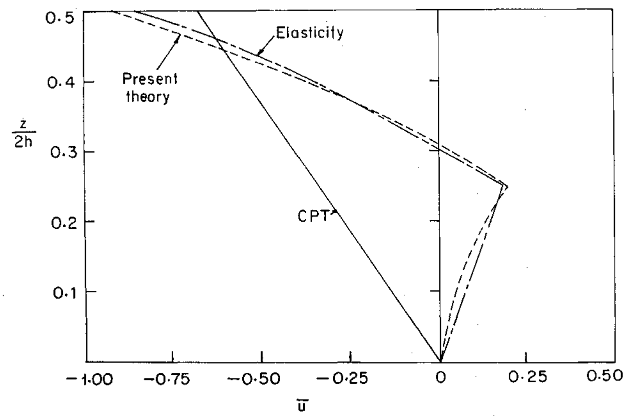

Fig. 4 Nondimensionalized in-plane deflection $\bar{u}$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

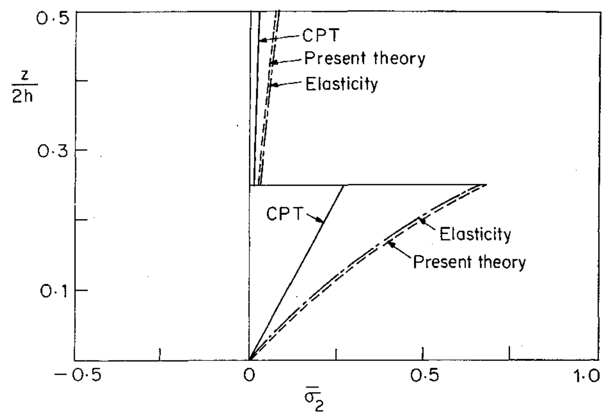

Fig. 7 Nondimensionalized in-plane normal stress $\bar{\sigma}_2$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

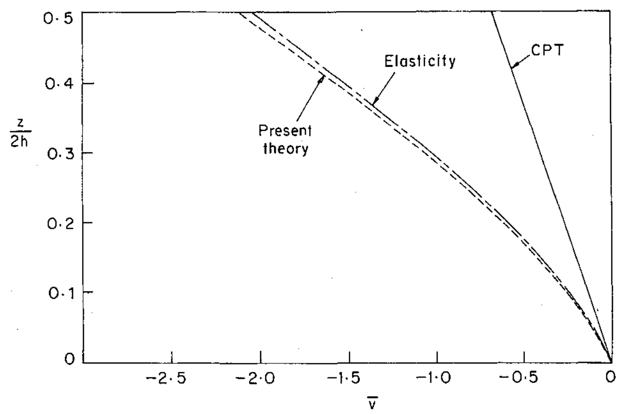

Fig. 5 Nondimensionalized in-plane deflection $\bar{v}$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

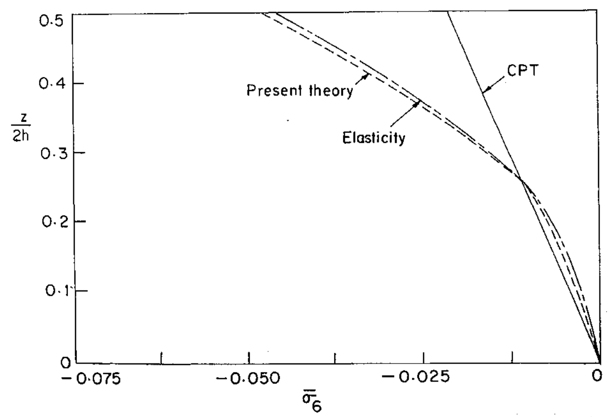

Fig. 8 Nondimensionalized in-plane shear stress $\bar{\sigma}_6$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

$$E_L = 25\ E_T, \qquad G_{LT} = 0.5\ E_T, \qquad G_{TT} = 0.2\ E_T$$

$$\nu_{LT} = \nu_{TT} = 0.25$$

For a $[0/90]_s$ laminate, $a/2h$ ratios of 4, 5, 6, 7.5, 10, 20, and 100 are considered. For a thick plate, $a/2h$ = 4, the thicknesswise distributions of displacements and stresses are presented in the nondimensionalized form as defined below:

$$\bar{w} = \frac{100\tilde{w}E_T}{2hq_oS^4} \tag{36a}$$

$$(\bar{u},\bar{v}) = \frac{100(\tilde{u},\tilde{v})\ E_T}{2hq_oS^3} \tag{36b}$$

$$(\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_6) = \frac{(\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_6)}{q_oS^2} \tag{36c}$$

$$(\bar{\sigma}_4,\bar{\sigma}_5) = \frac{(\tilde{\sigma}_4,\tilde{\sigma}_5)}{q_oS} \tag{36d}$$

where

$$S = a/2h \tag{36e}$$

For the same thick plate S = 4, $[0/90]_{NS}$ laminates are analyzed for N = 1, 2, 3, 4, 6, and 12. Also, E_L/G_{LZ} ratios of 10, 20, 30, 40, 50, and 80 are considered for the $[0/90]_s$ laminate.

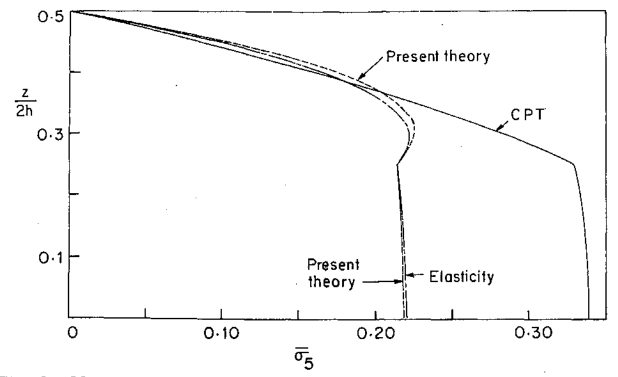

Fig. 9 Nondimensionalized transverse shear stress $\bar{\sigma}_5$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

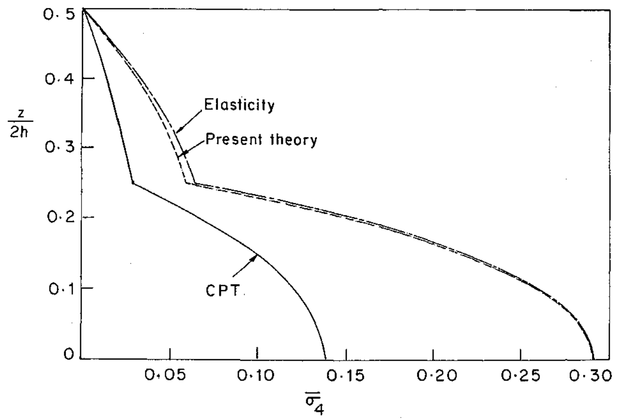

Fig. 10 Nondimensionalized transverse shear stress $\bar{\sigma}_4$ vs $z/2h$ (E_L/E_T = 25; $a/2h$ = 4).

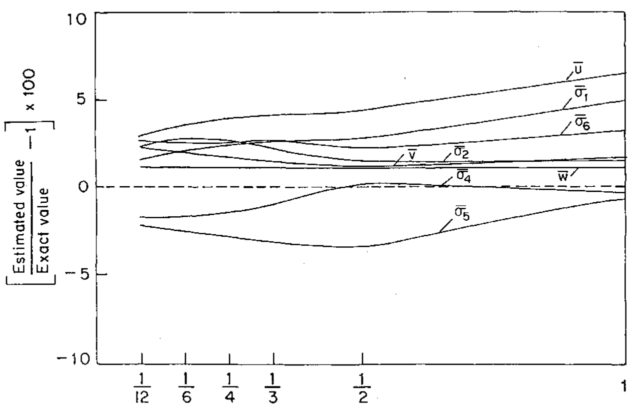

Fig. 11 Percentage errors in the estimates to displacements and stresses for $[0/90]_{NS}$ laminates vs $1/N$ (E_L/E_T = 25; $a/2h$ = 4).

For a $[0/90]_S$ laminate and E_L/E_T = 25, w vs $a/2h$ ratio is shown in Fig. 2. The agreement between the exact solution and the present procedure is excellent even for thick plates.

The estimates for in-plane stresses and transverse shear stresses are compared with the exact values in Table 2. The error is maximum in the estimation of σ_1 and is less than 5% even for S = 4.

The variations of percentage errors in the above estimates and the estimates for the maximum values of displacements with respect to $2h/a$ are shown in Fig. 3 (the corresponding errors in CPT are also shown in this figure). Errors in these estimates by the present procedure are generally found to increase with increasing values of $2h/a$ but relatively insensitive except for the estimates to u, σ_1, and σ_6. It is observed that the maximum errors are in the estimates of u and σ_1,

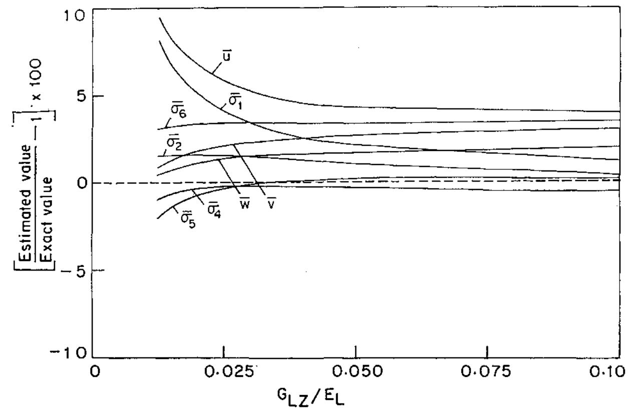

Fig. 12 Percentage errors in the estimates to displacements and stresses for $[0/90]_S$ laminate vs G_{LZ}/E_L ($a/2h$ = 4).

unlike in CPT, but they are far better estimates with errors less than 5% even for S = 5.

Thicknesswise distributions of displacements and stresses obtained by the present procedure are compared with the exact distributions in Figs. 4–10. It can be seen that distributions are in good agreement with the exact distributions and reflect all essential features of the exact solutions.

The percentage errors in the estimated physical quantities with increase in the number of plies in $[0/90]_{NS}$ laminates are shown in Fig. 11. The curves in this figure indicate that the number of plies in a laminate is not a critical parameter for the application of the present procedure.

Influence of the E_L/G_{LZ} ratio on the estimation of physical quantities is shown in Fig. 12. The magnitude of error increases with increase in the ratio E_L/G_{LZ} and becomes significant for thick and highly anisotropic plates.

Concluding Remarks

A new displacement-based model for the analysis of flexure of symmetric laminates has been developed. The displacements in the model, such as in group I models, are expressed in terms of reference-plane variables but take into account the ply-to-ply material discontinuities. Lamination effects are reflected, unlike in group II models, by a priori derived thicknesswise distributions. Numerical results obtained by the present theory are compared with the exact elasticity results for simply supported cross-ply square laminates subjected to sinusoidal loading. Errors in the estimates for displacements and stresses are generally found to increase with increasing values of $2h/a$ and E_L/G_{LZ} but relatively insensitive except for the estimates to u and σ_1 (σ_x). Out of these estimates, the error in the estimate to u_{max} is maximum and it is within 10% even for S = 4 and E_L/G_{LZ} = 80. The present model, such as in group I models, is quite suitable for finite element implementation.

Acknowledgments

The authors express their thanks to A. V. Krishna Murty for his valuable comments and keen interest shown during the course of the present investigation. The authors are grateful to Aeronautics Research and Development Board, Ministry of Defence, Government of India for providing financial assistance under Grant 481.

References

[1]Noor, A. K., and Burton, W. S., "Assessment of Shear-Deformation Theories for Multilayered Composite Plates," *Applied Mechanics Reviews*, Vol. 42, Jan. 1989, pp. 1–13.

[2]Reddy, J. N., "A Review of Refined Theories of Laminated Composite Plates," *Shock and Vibration Digest*, Vol. 22, No. 7, 1990, pp. 3–17.

[3]Reddy, J. N., "On the Generalization of Displacement-Based Laminate Theories," *Applied Mechanics Reviews*, Vol. 42, No. 2, Pt. 2, Nov. 1989, pp. S213–S222.

[4]Whitney, J. M., "Stress Analysis of Thick Laminated Composite and Sandwich Plates," *Journal of Composite Materials*, Vol. 6, Oct. 1972, pp. 426–440.

[5]Sun, C. T., and Whitney, J. M., "Theories for the Dynamic Response of Laminated Plates," *AIAA Journal*, Vol. 11, Feb. 1973, pp. 178–183.

[6]Srinivas, S., "A Refined Analysis of Composite Laminates," *Journal of Sound and Vibration*, Vol. 30, No. 4, 1973, pp. 495–507.

[7]Reddy, J. N., "A Generalization of Two-Dimensional Theories of Laminated Composite Plates," *Communications in Applied Numerical Methods*, Vol. 3, May/June 1987, pp. 173–180.

[8]Pagano, N. J., "Stress Fields in Composite Laminates," *International Journal of Solids and Structures*, Vol. 14, No. 4, 1978, pp. 385–400.

[9]Pagano, N. J., and Soni, S. R., "Global-Local Laminate Variational Model," *International Journal of Solids and Structures*, Vol. 19, No. 3, 1983, pp. 207–228.

[10]Ambartsumyan, S. A., *Theory of Anisotropic Plates*, Translated from Russian by T. Cheron, and edited by J. E. Ashton, Technomic, Stamford, CT, 1970, p. 76.

[11]Whitney, J. M., "The Effect of Transverse Shear Deformation on the Bending of Laminated Plates," *Journal of Composite Materials*, Vol. 3, July 1969, pp. 534–547.

[12]Valisetty, R. R., and Rehfield, L. W., "A Theory for Stress Analysis of Composite Laminates," *AIAA Journal*, Vol. 23, No. 7, 1985, pp. 1111–1117.

[13]Vijayakumar, K., and Krishna Murty, A. V., Iterative Modeling for Stress Analysis of Composite Laminates," *Composite Science and Technology*, Vol. 32, No. 3, 1988, pp. 165–181.

[14]Ren, J. G., "Bending Theory of Laminated Plate," *Composite Science and Technology*, Vol. 27, No. 3, 1986, pp. 225–248.

[15]Vijayakumar, K., Krishna Murty, A. V., Nagaraja, K. S., and Jawahar Reddy, K., "Higher-Order Theories for Estimation of the Interlaminar Stresses in Laminates," Dept. of Aerospace Engineering, Indian Inst. of Science, Bangalore, Rept. ARDB/STR/481/90/01, April 1990.

[16]Lee, K. H., Senthilnathan, N. R., Lim, S. P., and Chow, S. T., "An Improved Zigzag Model for the Bending of Laminated Composite Plates," *Composite Structures*, Vol. 15, No. 2, 1990, pp. 137–148.

[17]Pagano, N. J., "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," *Journal of Composite Materials*, Vol. 4, Jan. 1970, pp. 20–34.